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Optimal Error Bounds for Hermite Interpolation

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INTRODUCTION AND MAIN RESULTS

Let $u(x) \in c^4[0, h]$ be given; let $v_3(x)$ be the unique Hermite interpolation polynomial of degree $\leqslant 3$ satisfying

$$v_3(0) = u(0), \qquad v_3(h) = u(h), \qquad v_3'(0) = u'(0), \qquad v_3'(h) = u'(h). \quad (1.1)$$

Ciarlet, Schultz and Varga [2, Theorem 9] have obtained pointwise bounds for the error $e(x) = v_3(x) - u(x)$ and its derivatives in terms of $U = \max_{0 \leqslant x \leqslant h} |u^4(x)|$. Their bounds are

$$|e^{(k)}(x)| \leqslant \frac{h^k}{k!} \frac{(x(h-x))^{2-k}\, U}{(4-2k)!}, \qquad k = 0, 1, 2. \quad (1.2)$$

For $k = 0$, (1.2) is best possible, since equality holds for $u(x) = x^2(h-x)^2$, whose Hermite interpolation polynomial is $v = 0$. In 1967 Birkhoff and Priver [3] obtained following optimal error bounds on the derivatives $|e^{(k)}(x)|$ in terms of U.

THEOREM A (Birkhoff and Priver). *Let $u(x) \in c^4[0, 1]$. Then we have*

$$\begin{aligned} |e'(x)| &\leqslant \frac{Ux(1-x)(1-2x)}{12}, \qquad 0 \leqslant x \leqslant \frac{1}{3} \\ &\leqslant \frac{U}{96}\left[16x^3 - 105x^2 + 197x - 162 \right. \\ &\qquad \left. + \frac{66}{x} - \frac{13}{x^2} + \frac{1}{x^3}\right], \qquad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}, \end{aligned} \quad (1.3)$$

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$$|e''(x)| \leqslant \frac{U}{12(1-2x)^3}[48x^5 + 42x^4 - 100x^3$$

$$+ 54x^2 - 12x + 1], \qquad 0 \leqslant x \leqslant \frac{1}{3}$$

$$\leqslant \frac{U}{12}\left[\frac{1}{2} - 6\left(x - \frac{1}{2}\right)^2\right], \qquad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}, \tag{1.4}$$

$$|e'''(x)| \leqslant U\left[\frac{3}{16} + \frac{3}{2}\left(x - \frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^4\right], \qquad 0 \leqslant x \leqslant 1. \tag{1.5}$$

For $\frac{1}{2} \leqslant x \leqslant 1$ the bounds of $e^{(k)}(x)$ are given by

$$e^{(k)}(x) = e^k(1-x), \qquad k = 0, 1, 2, 3. \tag{1.6}$$

Further, from (1.2) ($k=0$), (1.3)–(1.6) the uniform error bounds are given by

$$|e^{(r)}(x)| \leqslant \alpha_r U, \qquad r = 0, 1, 2, 3; \qquad \alpha_0 = \frac{1}{4^2 4!},$$

$$\alpha_1 = \frac{3}{216}, \qquad \alpha_2 = \frac{1}{12}, \qquad \alpha_3 = \frac{1}{2}. \tag{1.7}$$

The object of this paper is to supplement the above theorem by proving the following theorems.

THEOREM 1. *Let $u(x) \in c^3[0, 1]$ and $v_3(x)$ be the unique, cubic Hermite interpolation polynomial satisfying* (1.1) (*with $h = 1$*). *Let $L = \max_{0 \leqslant x \leqslant 1} |f'''(x)|$. Then the following estimates are valid:*

$$|v_3(x) - u(x)| \leqslant \frac{4x^2(1-x)^3\, L}{3(3-2x)^2}, \qquad 0 \leqslant x \leqslant \frac{1}{2}$$

$$|v_3'(x) - u'(x)| \leqslant \frac{(2-3x)^3\, xL}{27(1-x)^2}, \qquad 0 \leqslant x \leqslant \frac{1}{3} \tag{1.8}$$

$$\leqslant \left[\frac{(2-3x)^3\, x}{27(1-x)^2} + \frac{(3x-1)^3\,(1-x)}{27x^2}\right] L, \qquad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}$$

$$|v_3''(x) - u''(x)| \leqslant 4x^2(1-x)^2\, L, \qquad \frac{1}{2} \leqslant x \leqslant \frac{2}{3} \tag{1.9}$$

$$\leqslant \left[4x^2(1-x)^2 + \frac{8}{27}\frac{(3x-2)^3}{(2x-1)^2}\right] L, \qquad \frac{2}{3} \leqslant x \leqslant 1, \tag{1.10}$$

$$|v_3'''(x) - u'''(x)| \leqslant \max_{0 \leqslant x,\, t \leqslant 1} |u'''(t) - u'''(x)|. \tag{1.11}$$

Moreover (1.8)–(1.10) remain valid if L is replaced by $\max_{0\leqslant x,t\leqslant 1}|f'''(t)-f'''(x)|$. Further, from (1.8)–(1.10) the uniform error bound is given by

$$|v_3^{(r)}(x)-u^{(r)}(x)|\leqslant \alpha_r L, \qquad \alpha_0=\frac{1}{96},$$

$$\alpha_1=\frac{13\sqrt{13}-46}{27}, \qquad \alpha_2=\frac{8}{27}, \qquad \alpha_3=2. \tag{1.12}$$

THEOREM 2. *Let* $u(x)\in c^2[0,1]$ *and* $v(x)$ *be the unique cubic Hermite interpolation polynomial satisfying* (1.1) (*with* $h=1$). *Let* $M=\max_{0\leqslant x\leqslant 1}|f''(x)|$. *Then we have*

$$|v_3(x)-u(x)|\leqslant\frac{4x^2(1-x)^2 M}{(1+2x)(3-2x)}, \qquad 0\leqslant x\leqslant 1, \tag{1.13}$$

$$|v_3'(x)-u'(x)|\leqslant\frac{x(6x^2-9x+4)^2 M}{6(1-x)}, \qquad 0\leqslant x\leqslant\frac{1}{3}$$

$$\leqslant\left[\frac{x(6x^2-9x+4)^2}{6(1-x)}+\frac{(1-x)(1-3x)^2}{6x}\right]M,$$

$$\frac{1}{3}\leqslant x\leqslant\frac{1}{2}. \tag{1.14}$$

Moreover, (1.13) *and* (1.14) *remain valid if we replace* M *by* $M_1=\max_{0\leqslant x,t\leqslant 1}|u''(x)-u''(t)|$. *Also,*

$$|v_3''(x)-u''(x)|\leqslant\frac{M_1}{3(1-2x)}[(1-3x)^2+(2-3x)^2], \qquad 0\leqslant x\leqslant\frac{1}{3}$$

$$\leqslant M_1, \qquad \frac{1}{3}\leqslant x\leqslant\frac{1}{2}. \tag{1.15}$$

Further, from (1.13)–(1.14) *the uniform error bound is given by* $(0\leqslant x\leqslant 1)$

$$|v_3^{(r)}(x)-u^{(r)}(x)|\leqslant\beta_r M, \qquad \beta_0=\tfrac{1}{16},\ \beta_1=0.251497657 \tag{1.16}$$

and

$$|v_3''(x)-u''(x)|\leqslant\tfrac{5}{3}M_1.$$

2. Preliminaries

It is known that the unique Hermite interpolation polynomial $v_3(x)$ satisfying (1.1) (with $h=1$) is given by

$$v_3(x)=u(0)(1-x)^2(1+2x)+u(1)\,x^2(3-2x)$$
$$+u'(0)\,x(1-x)^2+u'(1)\,x^2(x-1). \tag{2.1}$$

Using the well-known Peano theorem (3), we may write for $u(x)\in C^3[0,1]$

$$v_3^{(r)}(x)-u^{(r)}(x)=\int_0^1 G_3^{(r,0)}(x,t)\,u^{(3)}(t)\,dt, \qquad r=0,1,2, \tag{2.2}$$

where

$$G^{(i,j)}(x,t)=\frac{\partial^{i+j}G(x,t)}{\partial x^i\partial t^j}, \tag{2.3}$$

$$2G_3^{(0,0)}(x,t)=t(1-x)^2\,[2x-(1+2x)\,t], \qquad t\leqslant x$$
$$=x^2(1-t)[1+(2x-3)\,t], \qquad t\geqslant x, \tag{2.4}$$

$$G_3^{(1,0)}(x,t)=t(1-x)[1-3x+3tx], \qquad t\leqslant x$$
$$=x(1-t)[1-3t(1-x)], \qquad t>x, \tag{2.5}$$

and

$$G_3^{(2,0)}(x,t)=t[6x(1-t)+3t-4], \qquad t\leqslant x$$
$$=(1-t)[6xt+1-3t], \qquad t>x. \tag{2.6}$$

Similarly, on using the Peano theorem again [3], we may write for $u\in c^2[0,1]$

$$v_3^{(r)}(x)-u^{(r)}(x)=\int_0^1 u^{(2)}(t)\,G_2^{(r,0)}(x,t)\,dt, \qquad r=0,1, \tag{2.7}$$

$$v_3''(x)=-2\int_0^1 u''(t)[3x-2+3t(1-2x)]\,dt, \tag{2.7a}$$

$$v_3''(x)-u''(x)=-2\int_0^1 [u''(t)-u''(x)][3x-2+3t(1-2x)]\,dt, \tag{2.7b}$$

where

$$G_2^{(0,0)}(x,t)=(1-x)^2\,[x-t(1+2x)], \qquad t\leqslant x$$
$$=x^2[t(3-2x)-(2-x)], \qquad t\geqslant x, \tag{2.8}$$

$$G_2^{(1,0)}(x,t)=(1-x)[6tx+1-3x], \qquad t\leqslant x$$
$$=x[6t(1-x)-(4-3x)], \qquad t\geqslant x. \tag{2.9}$$

3. Proof of Theorem 1

Let $\frac{1}{2} \leqslant x \leqslant 1$ and note that

$$\alpha_0 = \alpha_0(x) = \frac{2x}{1+2x} \leqslant x; \qquad \beta_0 = \beta_0(x) = \frac{1}{3-2x} \leqslant x. \tag{3.1}$$

From (2.4) and (3.1) it follows that for $\frac{1}{2} \leqslant x \leqslant 1$

$$\begin{aligned}\int_0^x |G_3^{(0,0)}(x,t)|\,dt &= \frac{(1-x)^2\,(1+2x)}{2}\left[\int_0^{\alpha_0} t(\alpha_0 - t)\,dt + \int_{\alpha_0}^x t(t-\alpha_0)\,dt\right]\\ &= \frac{1}{3}\,\frac{x^3(3-3x+4x^3)(1-x)^2}{(1+2x)^2}.\end{aligned} \tag{3.2}$$

Again, on using (2.4) and (3.1) it follows that for $\frac{1}{2} \leqslant x \leqslant 1$

$$\int_x^1 |G_3^{(0,0)}(x,t)|\,dt = \frac{x^2(3-2x)}{2}\int_x^1 (1-t)(t-\beta_0)\,dt = \frac{x^3(1-x)^3}{3}. \tag{3.3}$$

From (3.2) and (3.3) we have

$$\int_0^1 |G_3^{(0,0)}(x,t)|\,dt = \frac{4x^3(1-x)^2}{3(1+2x)^2}, \qquad \frac{1}{2} \leqslant x \leqslant 1. \tag{3.4}$$

From symmetry arguments we also have

$$\int_0^1 |G_3^{(0,0)}(x,t)|\,dt = \frac{4x^2(1-x)^3}{3(3-2x)^2}, \qquad 0 \leqslant x \leqslant \frac{1}{2}. \tag{3.5}$$

From (3.4), (3.5) and (2.2) follow (1.8). Next we prove (1.9). First, we note that

$$\begin{aligned}\alpha_1 = \alpha_1(x) &= \frac{1}{3(1-x)} \geqslant x, \qquad 0 \leqslant x \leqslant \frac{1}{2};\\ \beta_1 = \beta_1(x) &= \frac{3x-1}{3x} \leqslant x \qquad \text{for } \frac{1}{3} \leqslant x \leqslant \frac{1}{2}.\end{aligned} \tag{3.6}$$

Let $0 \leqslant x \leqslant \frac{1}{3}$. Then, on using (2.5) and (3.6), we obtain

$$\begin{aligned}\int_0^x |G_3^{(1,0)}(x,t)|\,dt &= (1-x)\int_0^x (t(1-3x) + 3xt^2)\,dt\\ &= \frac{x^2(1-x)^2\,(1-2x)}{2}\end{aligned} \tag{3.7}$$

and

$$\int_x^1 |G_3^{(1,0)}(x,t)|\,dt = 3x(1-x)\left[\int_x^{\alpha_1}(1-t)(\alpha_1-t)\,dt + \int_{\alpha_1}^1 (1-t)(t-\alpha_1)\,dt\right]$$
$$= \frac{x(1-x)}{2}\left[2(1-\alpha_1)^3 + 2(1-x)^3 - 3(1-x)^2(1-\alpha_1)\right]. \tag{3.8}$$

Combining (3.7), (3.8), and using (3.6) we obtain for $0 \leqslant x \leqslant \frac{1}{3}$

$$\int_0^1 |G_3^{(1,0)}(x,t)|\,dt = \frac{(2-3x)^3 x}{27(1-x)^2}. \tag{3.9}$$

Next, consider $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$. Using (2.5), (3, 6) we have

$$\int_0^x |G_3^{(1,0)}(x,t)|\,dt = 3x(1-x)\left[\int_0^{\beta_1}(\beta_1-t)\,t\,dt + \int_{\beta_1}^x (t-\beta_1)\,t\,dt\right]$$
$$= x(1-x)\left[\beta_1^3 + \frac{3}{2}x^2(x-\beta_1)\frac{-x^3}{2}\right]. \tag{3.10}$$

Similarly

$$\int_x^1 |G_3^{(1,0)}(x,t)|\,dt$$
$$= 3x(1-x)\left[\int_x^{\alpha_1}(\alpha_1-t)(1-t)\,dt + \int_{\alpha_1}^1 (t-\alpha_1)(1-t)\,dt\right]$$
$$= x(1-x)\left[(1-\alpha_1)^3 - \frac{(1-x)^3}{2} + \frac{3}{2}(\alpha_1-x)(1-x)^2\right]. \tag{3.11}$$

On combining (3.10) and (3.11) we have $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$

$$\int_0^1 |G_3^{(1,0)}(x,t)|\,dt = \frac{x(1-x)}{27}\left[\frac{(3x-1)^3}{x^3} + \frac{(2-3x)^3}{(1-x)^3}\right]. \tag{3.12}$$

From (3.19), (3.12) and (2.2) follows (1.9). Now, we aim to prove (1.10). First, let $\frac{1}{2} \leqslant x \leqslant \frac{2}{3}$. Then from (2.6) it follows that

$$\int_0^1 |G_3^{(2,0)}(x,t)|\,dt = \int_0^x t[2(2-3x) + 3t(2x-1)]\,dt + \int_x^1 (1-t)[1 + 3t(2x-1)]\,dt$$
$$= 4x^2(1-x)^2. \tag{3.13}$$

Next we consider $\frac{2}{3} \leqslant x \leqslant 1$ and we note that

$$\alpha_2 = \alpha_2(x) = \frac{2(3x-2)}{3(2x-1)} \leqslant x. \tag{3.14}$$

Now, on using (2.6) and (3.14) we obtain

$$\begin{aligned}\int_0^x |G_3^{(2,0)}(x,t)|\,dt &= 3(2x-1)\int_0^x t\,|(\alpha_2 - t)|\,dt \\ &= 3(2x-1)\left[\int_0^{\alpha_2} t(\alpha_2 - t)\,dt + \int_{\alpha_2}^x t(t-\alpha_2)\,dt\right] \\ &= \frac{(2x-1)}{2}\,[2\alpha_2^3 + 2x^3 - 3\alpha_2 x^2].\end{aligned} \tag{3.15}$$

Further, on using (2.6) and (3.14) we obtain

$$\int_x^1 |G_3^{(2,0)}(x,t)|\,dt = \int_x^1 (1-t)[1+3t(2x-1)]\,dt = 2x^2(1-x)^2. \tag{3.16}$$

From (3.15), (3.16) we obtain for $\frac{2}{3} \leqslant x \leqslant 1$

$$\int_0^1 |G_3^{(2,0)}(x,t)|\,dt = 4x^2(1-x)^2 + \frac{8}{27}\frac{(3x-2)^3}{(2x-1)^2}. \tag{3.17}$$

From (3.13) and (3.17) we prove (1.10). It remains to prove (1.11).

For this purpose we note that

$$v_3'''(x) = 6\int_0^1 t(1-t)\,u'''(t)\,dt, \qquad 1 = 6\int_0^1 t(1-t)\,dt.$$

Therefore

$$v_3'''(x) - u'''(x) = 6\int_0^1 t(1-t)(u'''(t) - u'''(x))\,dt. \tag{3.18}$$

From (3.18), (1.11) follows at once. From

$$\int_0^1 G_3^{(r,0)}(x,t)\,dt = 0, \qquad r = 0, 1, 2, \tag{3.19}$$

it follows that for $r = 0, 1, 2$

$$v_3^{(r)}(x) - u^{(r)}(x) = \int_0^1 G_3^{(r,0)}(x,t)[u'''(t) - u'''(x)]\,dt.$$

Therefore

$$|v_3^{(r)}(x)-u^{(r)}(x)| \leqslant \max_{0\leqslant x,\, t\leqslant 1} |u'''(t)-u'''(x)| \cdot \int_0^1 |G_3^{(r,0)}(x,t)|\, dt.$$

Now, using (3.4), (3.5), (3.9), (3.12), (3.13) and (3.17), the same estimates hold in (1.8), (1.9) and in (1.10) provided L is replaced by $\max_{0\leqslant x,\, t\leqslant 1} |f'''(t)-f'''(x)|$. This proves Theorem 1.

The proof of Theorem 2 is similar to Theorem 1, so we omit the details.

4. Remarks

Theorems 1 and 2 are best possible. Let us consider the function

$$\begin{aligned} f_n(x) &= \left(1-\frac{2}{n}\right)x^2-\frac{4}{3}x^3+\frac{6n-4}{3n^2}x+\frac{4n-3n^2-2}{6n^2}, \\ &\qquad 0\leqslant x\leqslant \frac{1}{2}-\frac{1}{n} \\ &= \frac{n(2x-1)^4}{48}-\frac{(2x-1)^2}{4}+\frac{1}{12}, \qquad \frac{1}{2}-\frac{1}{n}\leqslant x\leqslant \frac{1}{2}+\frac{1}{n} \\ &= \left(1-\frac{2}{n}\right)(1-x)^2-\frac{4}{3}(1-x)^3+\frac{(6n-4)}{3n^2}(1-x) \\ &\quad +\frac{4n-3n^2-2}{6n^3}, \qquad \frac{1}{2}+\frac{1}{n}\leqslant x\leqslant 1. \end{aligned}$$

It is easy to see that $f_n(x)\in C^{(3)}[0,1]$ and $\max_{0\leqslant x\leqslant 1} |f_n'''(x)|=8$. If $v_3(f_n, x)$ denotes the corresponding cubic Hermite interpolation polynomial then one obtains

$$\left|f_n\left(\frac{1}{2}\right)-v_3\left(f_n,\frac{1}{2}\right)\right|=\frac{1}{96}-\frac{1}{24n^2}+\frac{1}{24n^3}.$$

Therefore (1.12) is best possible for $r=0$. Next, we will prove that (1.12) is best possible for $r=1$. Let us consider

$$\begin{aligned} g_n(x) &= \frac{x^3}{6}+\frac{A}{2}x^2, \qquad 0\leqslant x\leqslant \alpha-\frac{1}{n} \\ &= -\frac{n}{24}(x-\alpha)^4+\frac{1}{2}\left(\alpha-\frac{1}{2n}+A\right)(x-\alpha)^2+B(x-\alpha)+C, \\ &\qquad \alpha-\frac{1}{n}\leqslant x\leqslant \alpha+\frac{1}{n} \end{aligned}$$

$$= \frac{-x^3}{6} + (2\alpha + A)\frac{x^2}{2} + \left(\frac{1}{2} - 2\alpha - A\right)x + D,$$

$$\alpha + \frac{1}{n} \leqslant x \leqslant 1,$$

where

$$\alpha = \frac{\sqrt{13} - 1}{6}, \qquad \beta = \frac{5 - \sqrt{13}}{6}, \qquad A = \alpha^2 - 2\alpha + \frac{1}{2} + \frac{1}{3n^2},$$

$$B = \frac{\alpha^2}{2} + A\alpha - \frac{1}{6n^2}, \qquad C = \frac{\alpha^3}{6} + \frac{A}{2}\alpha^2 - \frac{1}{24n^3},$$

$$D = \frac{\alpha^3}{3} + \frac{\alpha}{3n^2}.$$

Again, it is easy to see that $g_n(x) \in C^{(3)}[0, 1]$, $\max_{0 \leqslant x \leqslant 1} |g_n'''(x)| = 1$.

$$\lim_{n \to \infty} |v_3'(g_n, \beta) - g_n'(\beta)| = \frac{13\sqrt{13} - 46}{27},$$

where β is defined above. This proves that (1.12) cannot be improved in the case $r = 1$ as well. Next, we consider

$$h_n(x) = \frac{x^3}{6} + \frac{x^2}{2}\left(-\frac{7}{18} + \frac{1}{2n}\right) + x\left(\frac{1}{6n^2} - \frac{1}{3n}\right) + A_1,$$

$$0 \leqslant x \leqslant \frac{2}{3} - \frac{1}{n}$$

$$= -\frac{n}{24}\left(x - \frac{2}{3}\right)^4 + \frac{5}{36}\left(x - \frac{2}{3}\right)^2 - \frac{1}{27}\left(x - \frac{2}{3}\right) - \frac{1}{27},$$

$$\frac{2}{3} - \frac{1}{n} \leqslant x \leqslant \frac{2}{3} + \frac{1}{n}$$

$$= \frac{-x^3}{6} + \left(\frac{17}{18} + \frac{1}{2n}\right)\frac{x^2}{2} - \left(\frac{4}{9} + \frac{1}{3n} + \frac{1}{6n^2}\right)x + A_2,$$

$$\frac{2}{3} + \frac{1}{n} \leqslant x \leqslant 1,$$

where

$$A_1 = \frac{1}{9n} - \frac{1}{9n^2} + \frac{1}{24n^3}, \qquad A_2 = \frac{8}{81} + \frac{1}{9n} + \frac{1}{9n^2} + \frac{1}{24n^3}.$$

Again $h_n(x) \in C^{(3)}[0, 1]$, $\max_{0 \leqslant x \leqslant 1} |h_n'''(x)| = 1$. Also note that

$$|v_3''(h_n, 1) - h_n''(1)| = \frac{8}{27} - \frac{2}{3n^2}.$$

Thus (1.12) is also best possible for $r = 2$ as well. In a similar way it can be shown that (1.12) cannot be improved for $r = 3$ as well. We omit the details. It may be worthwhile to mention that in the limit as $f_{n \to ^n \infty}(x)$, tends to a perfect spline function. Similar statement is also true for $g_n(x)$ and $h_n(x)$.

In an analogous way it can be shown that Theorem 2 is also best possible.

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